First-passage times in multi-scale random walks: the impact of movement scales on search efficiency

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An efficient searcher needs to balance properly the tradeoff between the exploration of new spatial areas and the exploitation of nearby resources, an idea which is at the core of *scale-free* Lévy search strategies. Here we study multi-scale random walks as an approximation to the scale-free case and derive the exact expressions for their mean-first passage times in a one-dimensional finite domain. This allows us to provide a complete analytical description of the dynamics driving the asymmetric regime, in which both nearby and faraway targets are available to the searcher. For this regime, we prove that the combination of only two movement scales can be enough to outperform both balistic and Lévy strategies. This two-scale strategy involves an optimal discrimination between the nearby and faraway targets, which is only possible by adjusting the range of values of the two movement scales to the typical distances between encounters. So, this optimization necessarily requires some prior information (albeit crude) about targets distances or distributions. Furthermore, we found that the incorporation of additional (three, four, ...) movement scales and its adjustment to target distances does not improve further the search efficiency. This allows us to claim that optimal random search strategies in the asymmetric regime actually arise through the informed combination of only two walk scales (related to the exploitative and the explorative scale, respectively), expanding on the well-known result that optimal strategies in strictly uninformed scenarios are achieved through Lévy paths (or, equivalently, through a hierarchical combination of multiple scales).

Search theory aims at identifying optimal strategies that help to promote encounters between a searcher and its target. Statistical physics approaches often identify searchers as random walkers, capitalizing on the idea of search as movement under uncertainty. The assumption that the searcher lacks any information about target locations leads to the fundamental question of how individual paths should be orchestrated to enhance random encounter rates [1–4]. Based on these assumptions, a proper measure of search efficiency is given by the Mean-First Passage Time (MFPT) of the random walker through the target location, a quantity which is also the focus of interest in many other areas of physics and science [5, 6]. Random search theory was spurred in the nineties as a result of a series of works suggesting that Lévy patterns (particularly, Lévy walks) can be optimal strategies for uninformed space exploration (see, e.g., [7–12]), and it has continually been developed since then.

Lévy walks can be defined as random paths composed of statistically identical flights whose length probability distribution decays asimptotically as $\sim |x|^{-\mu}$ (with $1 < \mu < 3$), where μ is frequently known as the *Lévy exponent*. So, lower values of the Lévy exponent comparatively imply a higher frequency of long flights. As a remarkable result, Viswanathan and colleagues [12] identified two different regimes associated to random search dynamics depending on whether targets are completely or uncompletely depleted after encounter, the so-called *destructive* and *non-destructive* dynamics, respectively. More recently [4, 13] it has been emphasized that these two dynamical regimes are much general and should be renamed and directly associated to searcher-to-target distances. In the *symmetric* regime targets are expected to occur at an average, characteristic distance from the searcher. For example, a *destructive* search dynamics tend to deploy targets locally and promote targets being faraway (on average) from the searcher, at least in low-density and homogeneous landscapes. A *symmetric* regime may also correpond to high target density scenarios where (on average) most targets can be assumed to be closeby. In the *asymmetric*, instead, a wide variety of searcher-to-target distances exist (i.e. heterogeneous landscapes), and both near and faraway targets may coexist in different proportions.

A more convenient understanding and interpretation of these regimes can be attained if linked to the general binomia exploitation-exploration [4]. According to it, three different scenarios should be distinguished: (i) those situations in which exploration is clearly preferred over exploitation (so a ballistic strategy, defined as a straight trajectory without changes of direction, is then trivially expected to be optimal). This is the case where revisiting areas is worthless and

the optimization requires performing displacements as long as possible without changing direction; a ballistic strategy is thus preferable. This happens if targets are uniformly distributed and can be fully depleted or if all targets are faraway (on average) from the searcher. (ii) those situations where exploitation prevails over exploration (so local, spatially bounded or diffusive search is optimal). This is the case of a searcher being nearby a set of targets (patch) that is never depleted. The random searcher always has the possibility to come back to the patch and the strategy of sticking around it is much preferable because no other targets are available in the landscape. And (iii) those situations where a true exploitation-exploration tradeoff emerges because the search necessarily requires the ability to reach both nearby and distant targets. For example if target distribution is patchy and while walking one can have nearby and faraway patches.

While the dynamics in the *symmetric* regime is thus straightforward to understand in terms of maximization (scenario i) or minimization (scenario ii) of the area explored, the details driving optimization in the *asymmetric* regime (scenario iii), in particular how movement scales determine search efficiency, have remained partially obscure to date [4, 13]. This is so because analytical methods for the determination of the MFPT (often valid just for Markovian processes) are difficult to extend to Lévy or other superdiffusive dispersal mechanisms. In the last years, much effort has been devoted to overcome this limitation. For instance, in [14] the asymptotic behaviour of the first-passage distribution of Lévy flights in semi-infinite media was obtained. Other authors have derived expressions and scaling properties of MFPTs for moving particles described either by Fractional Brownian Motion [15, 16] or fractional diffusion equations [17–19]. Finally, the alternative approach to approximate Lévy paths through an upper bound truncation (so that Lévy properties hold just over a specific set of scales) has been explored too [13]. But despite these advances, analytical arguments able to explain the different optimization dynamics observed in the *asymmetric* compared to the *symmetric* regime are still lacking.

Lévy or scale-free paths can be conveniently approximated through a combination of multiple scales [20]. This is tantamount to expressing power-law functions as a combination of exponentials [21, 22] or providing a Markovian embedding for Lévy stochastic processes [23–25]. Composite random walks and, more in general, Multi-Scale Random Walks (MSRW) have also emerged recently as an alternative to the presence of scale-free signatures in animal trajectories [21, 26]. It is not clear yet whether the emergence of multi-scaled movement behaviour in biology responds to exploratory behaviour tuned to uncertainty (Lévy as the limiting case) [4, 13], or else to informed behavioural processes linked to landscape through sensors and/or memory. It is thus important to understand how this multi-scaled behaviour should be coupled with other relevant landscape magnitudes like target distributions and searcher-to-target average distances.

Inspired by these ideas, in the present work we derive an exact analytical method for the determination of the MFPT of MSRWs as an approximation to the scale-free case. While the method proposed becomes increasingly complicated as more scales are considered, we show that 2-scale random walks can effectively resolve the explotation-exploration tradeoff emergent in the *asymmetric* regime by adjusting movement scales to target distances. Furthermore, the comparison between the 2-scale and the 3-scale random walk suggests that incorporating a third scale does not produce any advantage. Therefore, we conclude that an optimal random search strategy in the *asymmetric* regime consists on combining two informed movement scales that should approximately correspond to nearby/faraway target distances. Hence, an informed adjustment of movement scales improves search efficiency compared to any non-informed strategy (where scales are imposed at random). In the case of non-informed strategies, however, MSRWs aproximating the Lévy strategy are the best solution to solve exploitation-exploration tradeoffs.

DERIVATION OF THE MFPT

We consider for simplicity an isotropic random walk embedded in the one-dimensional finite domain $(0, L)$ with initial position x_0 and absorbing boundaries (so implicitly assuming that surrounding targets are located at $x = 0$ and $x = L$). We will choose $x_0 < L/2$ arbitrarily, so x_0 can be interpreted as the initial distance of the searcher to the nearest target. The searcher moves continuously with constant speed v and performs consecutive flights whose duration is distributed according to a multiexponential distribution function $\varphi(t)$ in the form

$$\varphi(t) = \sum_{i=1}^{n} w_i \varphi_i(t), \quad \varphi_i(t) = \tau_i^{-1} e^{-t/\tau_i}, \tag{1}$$

so yielding a n-scale MSRW characterized by the persistence times τ_i and their corresponding weights w_i, that satisfy the normalization condition $\sum_{i=1}^{n} w_i = 1$.

We now define $\rho_i(x, t; \boldsymbol{\rho}_0)$ as the probability that the walker starts at time t from x a single flight characterized by the distribution $\varphi_i(t)$ (in the following, using a particular distribution $\varphi_i(t)$ is termed as *being in state i*). The vector

$\boldsymbol{\rho}_0 = \delta(x - x_0)(\rho_{10}, \rho_{20}, \ldots, \rho_{n0})$ accounts for the set of initial conditions in all states, with ρ_{i0} the probability of being in state i at $t = 0$. Using this notation, the multi-scale (non-Markovian) walk gets reduced to a set of n Markovian states which satisfy (according to standard prescriptions of the Continuous-Time Random Walk [4]) the mesoscopic balance equations

$$\rho_i(x, t; \boldsymbol{\rho}_0) = w_i \sum_{k=1}^{n} \int_0^t \left(\frac{\rho_{k+} + \rho_{k-}}{2} \right) \varphi_k(t')dt' + \rho_{i0}\delta(x - x_0)\delta(t) \tag{2}$$

(for $i = 1, 2, \ldots, n$), where we have introduced the compact notation $\rho_{i\pm} \equiv \rho_i(x \pm vt', t - t'; \boldsymbol{\rho}_0)$. The corresponding probability that the walker, passing through x at time t, is performing at that instant a flight in state i is given by

$$P_i(x, t; \boldsymbol{\rho}_0) = \int_0^t \left(\frac{\rho_{i+} + \rho_{i-}}{2} \right) \tau_i\varphi_i(t')dt'. \tag{3}$$

Here we have used the relation $\int_t^\infty \varphi_i(t')dt' = \tau_i\varphi_i(t)$, valid for exponential distributions, which gives the probability that a single flight in state i will last at least a time t.

Due to the Markovian embedding used, the general propagator of the random walk in an infinite media can be written in the Laplace space (with s the Laplace argument) as $\sum_i^n P_i(x, s; \boldsymbol{\rho}_0)$ with the probability density for state i, $P_i(x, t; \boldsymbol{\rho}_0)$, given by a sum of n exponentials

$$P_i(x, s; \boldsymbol{\rho}_0) = \sum_{j=1}^{n} \alpha_{ij}(s)e^{-\beta_j(s)|x - x_0|/v}, \tag{4}$$

where α_{ij} and β_j are positive constants to be determined from the solution of the system (1-3). Hence, the solution in the interval of interest $(0, L)$ with periodic boundary conditions reads

$$\begin{aligned} Q_i(x, s; \boldsymbol{\rho}_0) &\equiv \sum_{m=-\infty}^{\infty} P_i(x + mL, s; \boldsymbol{\rho}_0) = \\ &= \sum_{j=1}^{n} \alpha_{ij}(s)\frac{e^{-\beta_j(s)(L - |x - x_0|)/v}e^{-\beta_j(s)|x - x_0|/v}}{1 - e^{-\beta_j(s)L/v}}. \end{aligned} \tag{5}$$

Finally, the exact MFPT can be computed from Eq. (5) by extending known methods for Markovian processes; in particular, we employ here the renewal method for velocity models [27, 28]. According to this, we define $f_i(t; \boldsymbol{\rho}_0)$ as the first-passage time probability rate for a walker through any of the boundaries while being in state i. The renewal property of Markovian processes allows then to write the recurrence relations

$$q_i(t; \boldsymbol{\rho}_0) = f_i(t; \boldsymbol{\rho}_0) + \sum_{k=1}^{n} \int_0^t f_k(t - t'; \boldsymbol{\rho}_0)q_i(t'; \boldsymbol{\rho}_k)dt', \tag{6}$$

where $q_i(t; \boldsymbol{\rho}_0)$ is defined as the probability rate with which the walker hits (not necessarily for the first time) the boundary at time t while being in state i. The term $q_i(t; \boldsymbol{\rho}_k)$ has the same meaning but for a walker starting its path at state k from the boundary (so with $x_0 = 0$). According to (6) the hitting rate $q_i(t; \boldsymbol{\rho}_0)$ gets divided into those trajectories for which this is the first hitting rate ($f_i(t; \boldsymbol{\rho}_0)$) plus those trajectories that hit the boundary for the first time at a previous time $t - t'$ in any of the possible n states (second term on the lhs of (6)). The total first-passage distribution of the MSRW will read then $f(t; \boldsymbol{\rho}_0) = \sum_{i=1}^{n} f_i(t; \boldsymbol{\rho}_0)$ (where the f_i's are to be determined from the system of equations (6)), and the general expression for the MFPT will be by definition [4]

$$\langle T \rangle = \lim_{s \to 0} \sum_{i=1}^{n} \frac{df_i(s; \boldsymbol{\rho}_0)}{ds}. \tag{7}$$

Then, to find a closed expression for $\langle T \rangle$ one just needs to express the hitting rates $q_i(t; \boldsymbol{\rho}_0)$ and $q_i(t; \boldsymbol{\rho}_k)$ in terms of the solutions of the random-walk (1-5). This is given, in analogy to previous works [28, 29], by

$$\begin{aligned} q_i(t; \boldsymbol{\rho}_0) &= \begin{cases} vQ_j(0, t; \boldsymbol{\rho}_0), & 0 < x_0 < L \\ vQ_j(0, t; \boldsymbol{\rho}_0) - \delta(t)/2, & x_0 = 0, x_0 = L; \end{cases} \\ q_i(t; \boldsymbol{\rho}_k) &= vQ_j(0, t; \boldsymbol{\rho}_k) - \delta(t)/2. \end{aligned} \tag{8}$$

Here clearly a different behaviour for the case when the walker starts from the boundaries is introduced by convenience to make explicit that the walker cannot get trapped by the boundary immediately at $t = 0$, but hittings are only possible for $t > 0$. In the following we study how the different scales considered in the MSRW contribute to the search efficiency as a function of the two prominent spatial scales present in the problem (i.e. x_0 and L).

1-scale case ($n = 1$)

The MSRW scheme described above reduces trivially in this case to a classical Correlated Random Walk (see, e.g., [4, 30, 31]) for which the free propagator (Equation (4)) reads

$$P_1(x, s; \boldsymbol{\rho}_0) = \frac{1}{2v}\sqrt{\frac{s + \tau_1^{-1}}{s}}\exp\left[\sqrt{s\left(s + \tau_1^{-1}\right)}|x - x_0|/v\right] \tag{9}$$

Using the derivations in Equations (5-8), the MFPT in (7) yields the exact expression obtained by Weiss [32] thirty years ago

$$\langle T \rangle = \frac{L}{2v} + \frac{x_0(L - x_0)}{v^2\tau_1}. \tag{10}$$

So, assuming that x_0, L are fixed by the external or environmental conditions, we observe that the search optimization of the 1-scale random walk turns out to be trivial: faster searches (i.e. larger values of v) and straighter trajectories (i.e. $\tau_1 \to \infty$) will monotonically reduce the search time. In particular, note that for $\tau_1 \to \infty$ one recovers the result $\langle T \rangle = L/2v$, which coincides with the result for a ballistic strategy. It is clear then that in 1-scale random walks the exploration-exploitation tradeoff (L vs x_0) is always trivially optimized through a ballistic strategy (in agreement with the results in [13]). As we shall see in the following, at least 2 scales are necessary in the random walk to observe such effects.

2-scales case ($n = 2$)

The exact analytical solution for this case can still be found easily, albeit the general expression for the MFPT obtained is cumbersome; details are provided in the Supplementary Information (SI) file. A first survey on this solution (which was implemented in MAPLE) allows us to observe that for *large* values of x_0 the balistic-like strategy (i.e. $\tau_1 \to \infty$, $\tau_2 \to \infty$) is again the one which minimizes $\langle T \rangle$. However, for *small* x_0 values we now find the emergence of an asymmetric regime in which the optimal is attained for one of the two scales (either τ_1 or τ_2) being much larger than the time L/v required to cover the domain, with the other scale exhibiting a smaller value. The threshold at which this transition occurs (so, the value of x_0 for which the optimum $\langle T \rangle$ becomes smaller than $L/2v$) turns out to be $x_0 \approx 0.105L$, a value which is confirmed by random-walk simulations too.

At the sight of these results, we will focus now our interest in providing some limit expressions which can help us to understand how this transition occurs and how the system behaves in the *asymmetric* regime. First we note that, in solving the exploration-exploitation tradeoff, the exploration part will be always optimized through flights much longer than the typical time to cover the whole domain, which explains why one of the two scales (say, τ_1) should be expected to be as large as possible, in particular $\tau_1 \gg L/v$. Regarding the second scale, the exploitation side of the tradeoff (corresponding to exploring the surrounding area searching for nearby targets) should intuitively benefit from choosing a scale of the order of $\tau_2 \sim x_0/v$, the time required to reach the nearest target. Scales much larger than this would promote exploration instead of exploitation, while scales much smaller would lead to an unnnecessary overlap of the searcher's trajectory around its initial position [13]. Since the *asymmetric* regime must emerge necessarily from the asymmetric condition $x_0 \ll L$ we can thus consider that this second scale should satisfy $\tau_2 \ll L/v$.

Taking the two limits ($\tau_1 v/L \to \infty$ and $\tau_2 v/L \to 0$) into account, our general solution for the MFPT reduces to

$$\langle T \rangle = \frac{L}{2v} + \frac{\tau_2(1 - w_1)}{w_1}\left(1 - \frac{1 + \frac{Lw_1}{2v\tau_2}}{1 + \sqrt{w_1}}\exp\left[\frac{-\sqrt{w_1}x}{v\tau_2}\right]\right). \tag{11}$$

Visual inspection of this expression already shows that values of the MFPT below the ballistic threshold $\langle T \rangle = L/2v$ can be now obtained for appropriate combinations of x_0, τ_2 and w_1. In particular, in the limit when $x_0 \to 0$ the previous expression gets minimized for the value

$$\tau_2^* = \sqrt{Lx_0w_1/2v^2} \tag{12}$$

where we use the asterisk to denote values that are *optimal*. After minimizing $\langle T \rangle$ also with respect to w_1 we find that the global optimum of the MFPT corresponds to

$$\tau_2^* = \frac{1}{v}\sqrt{\frac{x_0^2(L + \sqrt{8Lx_0})}{L - 8x_0}}, \quad w_1^* = \frac{2x_0(L + \sqrt{8Lx_0})}{L(L - 8x_0)}. \tag{13}$$

Now, in the limit $L \to \infty$ we observe that $\tau_2^* \to x_0/v$ and $w_1^* \to 2x_0/L$. Altogether, these results provide a clear and simple description of the search dynamics in the *asymmetric* regime for 2-scale MSRWs which confirms our discussion above. The optimum strategy in the *asymmetric* regime will combine a very large scale $\tau_1 \gg L/v$ (for exploration purposes) with a shorter scale τ_2 of the order of x_0/v (for better exploitation of the nearest target). It is particularly interesting that the optimal weight w_1^* must be rather small, so the searcher just needs occasional ballistic flights while spending the rest of the time searching intensively its surroundings. So, the optimal strategy does not consist just on an appropriate choice of the scales τ_1 and τ_2 but also on using them in an adequate proportion.

Figures 1 and 2 show the comparison between random-walk simulations (symbols) and our method, both for the exact case (solid lines) and for the approximations (11-13) (dotted lines). Note in Figure 1 that the optimum value of the MFPT clearly improves (specially for x_0/L very small) the value obtained for a ballistic strategy or a $\mu = 2$ Lévy walk strategy (dashed and dashed-dotted horizontal lines, respectively), so revealing that an appropriate combination of only two move length scales can be actually more efficient than a scale-free strategy. The range of validity of the approximated results (11-13) is also shown in the plots, as well as the scaling $\tau_2^* \sim \sqrt{w_1}$ derived in Eq. 12 (see Figure 2).

Despite finding a set of combinations of τ_2 and w_1 outperforming both Lévy and ballistic strategies, these results show that it is necessary for the searcher to have some information about the domain scales (i.e. x_0 and L) in order to fine-tune search and get effective strategies. Without this knowledge Lévy or ballistic patterns look as robust strategies, that could be even more effective than searching with badly adjusted movement scales, as suggested by the comparison in Figure 1. This fact is also confirmed when observing the dependence of $\langle T \rangle$ on τ_2 (Figure 3, see also Figure S1 in the SI) in order to assess the range width at which τ_2 and w_1 lead to optimality. In Figure 3 we provide the results of our exact solution for different values of x_0 and different weights w_1 (here the approximated results and simulations are not shown in order to facilitate understanding). In accordance to our analytical results above, we observe that values of w_1 close to $2x_0/L$ minimize the MFPT. So, there are certain values of w_1 for which the MFPT becomes lower than the balistic value $L/2v$ (but we we stress that the most critical parameter for getting below this threshold is clearly x_0). Actually, for the two upper panels (which correspond to $x_0/L = 0.005$ and $x_0/L = 0.01$) we observe that any choice of τ_2 and w_1 would result in a better (or as good as) performance than a balistic strategy, while the region where the Lévy strategy is outperformed is relatively small.

We stress finally that we have carried out studies, both analytically and numerically, for the case when the initial position x_0 is not fixed but is distributed according to an exponential or a Gaussian distribution, so a range of x_0 values is allowed (results not shown here). The results for all these cases coincide qualitatively with those reported above, so whenever values $x_0 < 0.105L$ are predominant the *asymmetric* regime is recovered. This confirms that the emergence of the *asymmetric* regime is not an artefact caused by the choice of a fixed initial condition, in agreement with recent numerical studies [33].

3-scales case ($n = 3$)

Provided that the initial time to reach any of the targets is given by the two timescales x_0/v and $(L - x_0)/v$, it could be intuitively expected that these are the only ones necessary to reach an optimal strategy. To check this we have solved analytically the 3-scale case (see SI) and, given that the expression obtained is extremely cumbersome, we have used Markov Chain Monte Carlo algorithms in order to determine numerically the global minimum of the MFPT as a function of the parameters τ_i and w_i. The results so obtained are conclusive and confirm the idea that indeed only two scale are needed to minimze the MFPT. We find that for large values of x_0 the optimal strategy is again ballistic-like (so only displacements with $\tau_i \gg L/v$ should be performed in order to minimize $\langle T \rangle$). Instead, for x_0 small enough the optimum arises through the combination of only two scales which coincide with those found for the optimal 2-scale case; this means that two of the three scales involved (say, τ_1 and τ_2) will eventually have the same value after minimization. Even more surprising is that when the initial condition is governed by two different scales (by combining two different values of x_0, each with a given probability) the optimum still corresponds to a 2-scale random-walk; in this case the optimum value τ_2^* is in between the optimum values that one would find for each of the two x_0 values alone. Further studies are thus needed to confirm to what extent the combination of only

two scales is universally robust and effective enough, independently of the number of prominent spatial scales present in the domain; this point will be the focus of a forthcoming work.

DISCUSSION

The main result extracted from the theoretical analysis reported here is that MSRWs with only two movement characteristic scales can represent a mathematical optimum (in terms of MFPT minimization) for random search strategies. This has been proved by checking that additional scales do not allow to improve the optimum achieved for 2 scales. The optimal solution outperforms both ballistic and Lévy strategies but only for specific intervals of the characteristic parameters τ_i and w_i which depend on the characteristic scales of the domain (namely, x_0 and L). In particular, the global optimum turns out to be given by $\tau_1 \gg L/v$ and $\tau_2 \approx 2x_0/v$, which can be intuitively justified in terms of optimizing the tradeoff between exploring for faraway targets and exploiting nearby resources. While the theoretical analysis provided here has been restricted to the one-dimensional case (for which an exact solution for the MFPT is attainable), we think that these arguments are generally valid and so we expect them to hold in higher dimensions, and probably in more complicated situations as for instance in biased searches [6] too.

In the context of animal foraging, the fact that fine-tuned 2-scale random-walks outperform Lévy walks represents a convenient extension of the Lévy flight paradigm from the completely uninformed scenario to that in which domain scales are (partially or completely) available to the organism. In the uninformed case, where the characteristic scales of the search problem are unknown, a scale-free strategy represents a convenient (albeit sub-optimal) solution. However, in cognitive systems search optimization programs should be adjustable on the basis of accumulated evidence. As we show here, 2-scale walks would be optimal provided that the searcher has previous available information (at least some crude guess) about the values of the scales x_0 and L. Let us stress that we are considering that such prior guess about target distances is limited (by the searcher cognitive capacity) or not informative enough (e.g. landscape noise, insufficient cumulative evidences) to set up a deterministic search strategy; so, the random-walk hypothesis is still meaningful. Accordingly, as more and more information about the domain scales becomes integrated by the searcher we should observe a tendency towards a reduction (and an adjustment) in the number (and magnitude) of movement scales used, respectively. This process should go on up to the point where barely one or two scales would persist. Furthermore, we note that for the extreme case of perfectly informed (deterministic) walkers no characteristic search scales at all would be necessary since in that case the search process is plainly directed towards the target.

Our results add then some new dilemmas and perspectives on the fundamental problem of what biological scales could be relevant in terms of a program driving animal paths to enhance foraging success. Within the uninformed scenario, Weierstrassian Walks involving a relatively low number of scales in a geometric progression have been proposed as an efficient mechanism to implement Lévy-like trajectories [21]. This itself builds on the more general idea of reproducing power-law paths through a hierarchical family of random walks [20]. These Weierstrassian Walks provide, due to its relative simplicity, a promising approach to bring together the ideas from the Lévy flight paradigm and those from MSRWs [26], although we stress that many alternative Markovian embeddings for power-laws do exist in the literature [22, 23]. Within this context, the existence of a correlation between the number of movement scales and the informational gain we suggest here may pose new challenges for experimentalists and data miners. For example, provided that we can conveniently interpret animal trajectories in terms of a combination of scales, can we infer something about the informational capacity of the individual from the number of scales observed and from the relation between their values? How can we differentiate informationally-driven scales from the internally-driven ones? While we are not yet in position to provide a definite answer to these questions, we expect that the ideas provided in this work can stimulate research in this line and can assist experimentalists towards new experimental designs for a better understanding of the interplay between animal foraging, landscape scales, and information processing.

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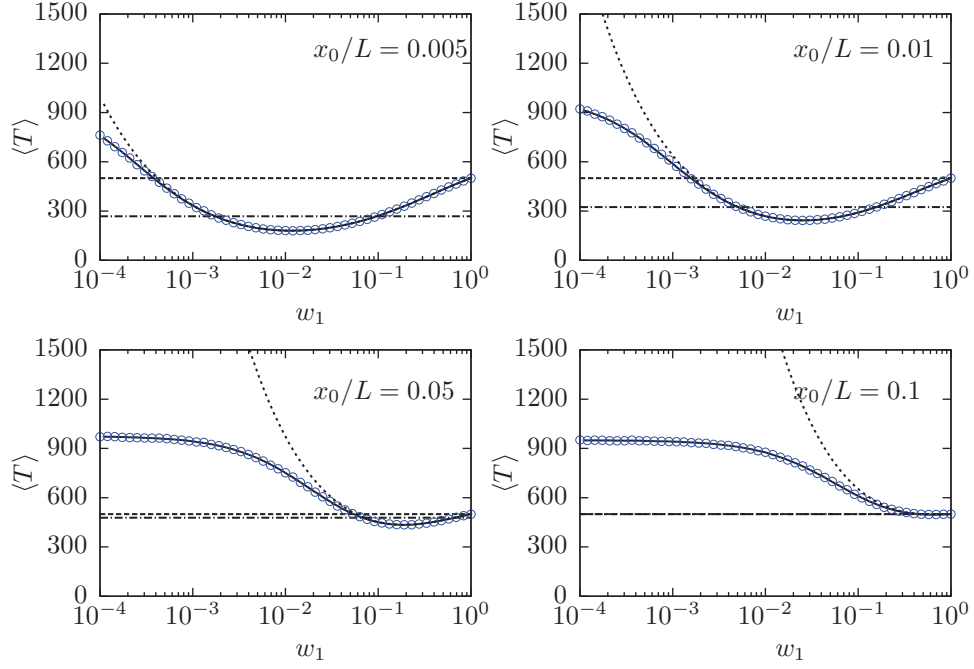
FIG. 1: MFPT for a 2-scale random walker with $L = 1000$, $v = 1$, $\tau_1 = 10^3 L/v$ and $\tau_2 = x_0/v$, and for different initial conditions. The plot shows the exact analytical solution (solid lines), random-walk simulations averaged over 10^6 realizations (circles) and the $L \to \infty$ approximation given by Eq. (11) (dotted lines). The values obtained for ballistic and $\mu = 2$ Lévy strategies are also given for comparison (dashed and dashed-dotted horizontal lines, respectively).
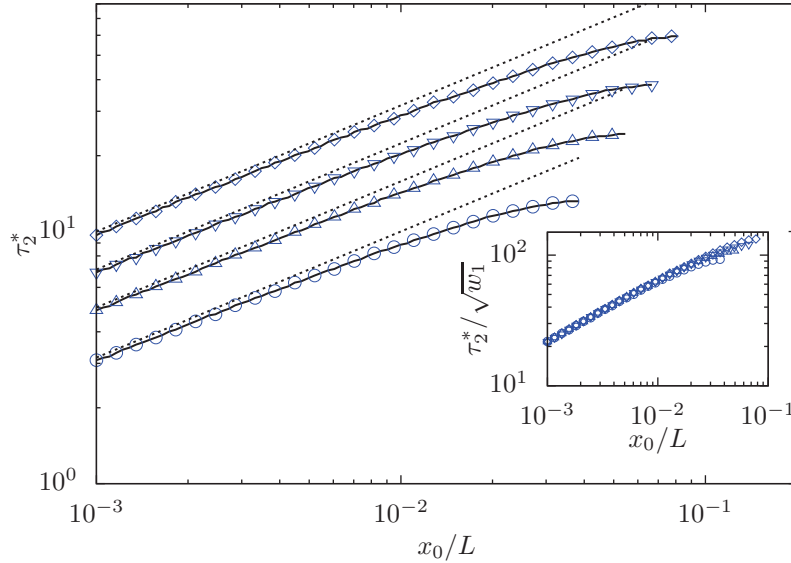

FIG. 2: Optimal persistence predicted by the exact analytical solution (solid lines) and the approximated expression $\tau_2^* = \sqrt{L x_0 w_1 / 2 v^2}$ (dotted lines) in comparison to random-walk simulations (symbols). Different values of the weight w_1 are shown: $w_1 = 0.02$ (circles), 0.05 (triangles), 0.1 (inverted triangles) and 0.2 (diamonds). Full symbols denote the x_0/L values above which the *symmetric* regime appears and so there is no optimal persistence τ_2^*. Inset: The same results are shown with $\tau_2^*/\sqrt{w_1}$ in the vertical axis. The collapse observed confirms the scaling $\tau_2^* \sim \sqrt{w_1}$ analytically derived.
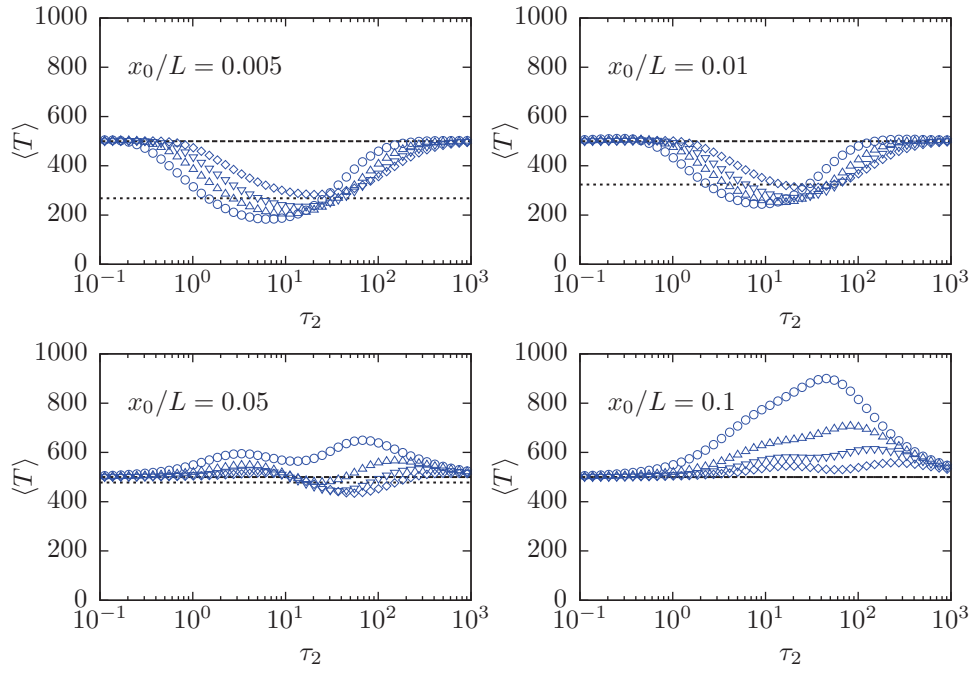
FIG. 3: MFPT for a 2-scale random walker with $L = 1000$, $v = 1$ and $\tau_1 = 10^3 L/v$ with different initial conditions. The plot shows the exact analytical solution for different values $w_1 = 0.02$ (circles), 0.05 (triangles), 0.1 (inverted triangles) and 0.2 (diamonds). The values obtained for ballistic and $\mu = 2$ Lévy strategies are also given for comparison (dashed and dotted lines, respectively).